Chapter 11 Study Notes

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Vectors and the Geometry of Space

1. The magnitude of a vector: $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

- CalcView: Section 1, Exercise 21 Video.

2. The dot product: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

- CalcView: Section 3, Exercise 3 Video.
- CalcView: Section 3, Exercise 19 Video.

3. The cross product: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

- CalcView: Section 4, Exercise 7 Video.

4. Normalize a vector: $\mathbf{u} = \dfrac{\mathbf{v}}{||\mathbf{v}||}$

- CalcView: Section 1, Exercise 35 Video.
- **OBJECTIVE:** Given **v**, find unit vector **u** in the same direction. Then verify **u** has length of one.
- **STEPS:**
  1) Write vector in component form.
  2) Divide each **v** component by the magnitude.
  3) Use formula above.
  4) To verify length of one, take magnitude of **u**.

5. Angle between vectors: $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||}$

- CalcView: Section 3, Exercise 13 Video.
- **OBJECTIVE:** Given **u** and **v**, find the angle between them.
- **STEPS:**
  1) Write vectors in component form.
  2) Take dot product of **u** and **v**.
  3) Take magnitude of **u** and **v**.
  4) Use formula above.

6. Projection: $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2}\right)\mathbf{v}$

- CalcView: Section 3, Exercise 37 Video.
- **OBJECTIVE:** Given **u** and **v**, find (1) projection of **u** onto **v**, and (2) vector component of **u** orthogonal to **v**.
- **STEPS:**
  1) Write vectors in component form.
  2) $\mathbf{u} = \mathbf{w_1} + \mathbf{w_2}$.
  3) Find $\mathbf{w_1}$ = projection of **u** onto **v**:
     a) Take dot product of **u** and **v**.
     b) Take magnitude squared of **v**.
     c) Use formula above.
  4) Find vector component of **u** orthogonal to **v**, which is $\mathbf{w_2} = \mathbf{u} - \mathbf{w_1}$.

7. Direction Cosines: $\cos\alpha = \frac{v_1}{||\mathbf{v}||} \qquad \cos\beta = \frac{v_2}{||\mathbf{v}||} \qquad \cos\gamma = \frac{v_3}{||\mathbf{v}||}$

- CalcView: Section 3, Exercise 31 Video.

8. Volume of a parallelepiped: $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

- CalcView: Section 4, Exercise 35 Video.
- **OBJECTIVE:** Use *triple scalar product* to find volume of parallelepiped.
- **STEPS:**
  1) Write vectors in component form.
  2) Write 3x3 matrix using vector components.
  3) Use first row components of matrix as coefficient for each of the three cross-products.
  4) Take absolute value of result.

9. Distance between two points: $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- CalcView: Section 2, Exercise 25 Video.

10. Midpoint: $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2}\right)$

- CalcView: Section 2, Exercise 33 Video.

11. Equation of a Sphere: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

- CalcView: Section 2, Exercise 39 Video.
- **OBJECTIVE:** Given endpoints of diameter of a sphere, write standard equation of the sphere.
- **STEPS:**
  1) Find center first, then radius.
  2) Find center by using the Midpoint Formula.
  3) Find radius by using Distance Formula and multiplying it by 1/2.
  4) Write equation and leave radius computed with a square over it.

12. Parametric Equation of a Line: $x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$

- CalcView: Section 5, Exercise 9 Video.
- **OBJECTIVE:** Find parametric set of equations, then symmetric set of equations, of the line passing through a given point and parallel to a given vector **v**. (Write direction numbers as integers).
- **STEPS:**
  1) Write vector in component form. This is the *Direction Vector*, **v** = <a, b, c>.
  2) Find parametric equations:
     a) $x = x_0 + at$
     b) $y = y_0 + bt$
     c) $z = z_0 + ct$
     d) Where,
        i. x_0, y_0, z_0 are coordinates of given point.
        ii. a, b, c are direction numbers from direction vector.
  3) Find symmetric equations:
     a) Use Formula 13 below and substitute known values for x_0, y_0, z_0 and a, b, c.
     b) If get 0 in denominator, say "not possible for symmetric equations".

- CalcView: Section 5, Exercise 15 Video.
- **OBJECTIVE:** Given two points P and Q, find parametric set of equations, then symmetric set of equations, of the line passing through the points. (Write direction numbers as integers).
- **STEPS:**
  1) Find vector PQ to use as the *Direction Vector*, **v** = <a, b, c>.
  2) Find parametric equations:
     a) $x = x_0 + at$
     b) $y = y_0 + bt$
     c) $z = z_0 + ct$
     d) Where,
        i. x_0, y_0, z_0 are coordinates of either point.
        ii. a, b, c are direction numbers from direction vector.
  3) Find symmetric equations:
     a) Use Formula 13 below and substitute known values for x_0, y_0, z_0 and a, b, c.
     b) If get 0 in denominator, say "not possible for symmetric equations".

13. Symmetric Equation of a Line: $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

- CalcView: Section 5, Exercise 9 Video. (See above)
- CalcView: Section 5, Exercise 15 Video. (See above)

14. Standard Form of a Plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

- CalcView: Section 5, Exercise 39 Video.
- **OBJECTIVE:** Given a point and a perpendicular vector **n**, find equation of the plane.
- **STEPS:**
  1) a, b, c, are components of a normal vector (perpendicular to plane).
  2) x_0, y_0, z_0 are coordinates of a point that lies on the plane.
  3) Substitute into Standard Form above.
  4) To check, substitute given point into obtained equation of the plane to ensure equals 0.

- CalcView: Section 5, Exercise 47 Video.
- **OBJECTIVE:** Given three points P, Q, R, find equation of the plane.
- **STEPS:**
  1) To find equation of the plane, need:
     a) A point on the plane. Can choose any of the three.
     b) A normal vector **n** to the plane.
        i. Find two vectors PQ and PR in component form.
        ii. Take cross-product of PQ and PR.
        iii. Normal vector **n** gives a, b, c.
  2) Substitute into Standard Form above.
  3) To check, substitute given points into obtained equation of the plane to ensure equals 0.

15. Distance between a Point and a Plane: $D = ||\text{proj}_{\mathbf{n}} \overrightarrow{PQ}|| = \dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{||\mathbf{n}||}$

- CalcView: Section 5, Exercise 87 Video.
- **OBJECTIVE:** Given a point Q (not on the plane) and a plane, find the distance between Q and the plane.
- **STEPS:**
  1) Need a point P on the plane. Find that point by using equation of the plane and set two of the variables equal to zero and solve for third variable.
  2) Find direction vector PQ using component-wise subtraction.
  3) Find normal vector **n** <a, b, c> by using variable coefficients of the equation of the plane.
  4) Use formula above to find:
     a) The dot product between PQ and **n**.
     b) The magnitude of **n**.

16. Distance between a Point and a Line: $D = \frac{||\overrightarrow{PQ} \times \mathbf{u}||}{||\mathbf{u}||}$

- CalcView: Section 5, Exercise 95 Video.
- **OBJECTIVE:** Given a point Q in space and a line represented by parametric equations, find the distance between them.
- **STEPS:**
  1) Need four items to substitute into the formula above:
     a) Q = point in space (not on the line) is given.
     b) P = point on the line.
        i. Set t=0 in each parametric equation to obtain the values of x, y, z, the point on the line.
     c) Find vector PQ using component-wise subtraction.
     d) Find **u**, the direction vector <a, b, c> that is parallel to the line.
        i. Get a, b, c from the coefficients of t in the parametric equations, where a corresponds to the x-term, b to the y-term, and c to the z-term.
        ii. Write **u** in component form.
  2) Find cross-product PQ x **u**.
  3) Find magnitude of the cross-product PQ x **u**.
  4) Find magnitude of **u**.

Chapter 11 Objectives

Section 1

Show vectors **u** and **v** are equivalent.
- CalcView: Exercise 5 Video.
- **STEPS:**
  1) Find magnitude of **u** and **v**.
  2) Find slope of **u** and **v**.

Given initial and terminal points, (1) sketch directed line segment, (2) write the vector in component form, (3) write the vector as a linear combination of unit vectors **i** and **j**, and (4) sketch vector with initial point at origin.
- CalcView: Exercise 11 Video.

Given magnitude of **v**, find **v** in the direction of **u.**
- CalcView: Exercise 47 Video.
- **STEPS:**
  1) Find magnitude of **u**.
  2) Divide vector **u** by the magnitude of **u** to normalize **u** for length of one in same direction as **u**. Name it vector **w**.
  3) Multiply magnitude of **v** with vector **w**.

Find the component form of **v** given its magnitude and direction (angle theta).
- CalcView: Exercise 49 Video.
- **STEPS:**
  1) Multiply magnitude of **v** with <cos(theta), sin(theta)>

Section 2

Given two points, find (1) component form of **v**, (2) magnitude of **v**, and (3) unit vector in direction of **v**.

- CalcView: Exercise 51 Video.

Determine if two vectors are parallel.

- CalcView: Exercise 63 Video.

Use vectors to determine if three given points are collinear.

- CalcView: Exercise 67 Video.

Given vector **v**, find unit vector in (1) direction of **v**, (2) opposite direction of **v**.

- CalcView: Exercise 79 Video.

Section 4

Given **u** and **v**, find a unit vector that is orthogonal to **u** and **v**.

- CalcView: Exercise 15 Video.

Given four points, show they are vertices of a parallelogram and then find the area.

- CalcView: Exercise 23 Video.

Find *triple scaler product.*

- CalcView: Exercise 31 Video.

Section 5

Given two planes, find (1) the angle between the planes, and (2) a set of parametric equations for the line of intersection.

- CalcView: Exercise 65 Video.

Given equation of a plane, find its three intercepts.

- CalcView: Exercise 79 Video.

Given two planes, (1) verify that the planes are parallel, and (2) find the distance between the planes.

- CalcView: Exercise 91 Video.

Section 6

Describe and sketch the surface generated by an equation.

- CalcView: Exercise 11 Video.

Classify and sketch the quadratic surface given by an equation.

- CalcView: Exercise 17 Video.

Find the equation for the surface of revolution formed by taking an equation in the yz-plane and revolving that around the y-axis.

- CalcView: Exercise 31 Video.

Find the generating curve given the equation of a surface of revolution.

- CalcView: Exercise 37 Video.