Properties of Logarithmic Functions

DEFINITION Logarithmic Function with Base *a*

The **logarithmic function with base *a***, where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ (read as "y is the logarithm with base a of x") and is defined by

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

- Domain of the logarithmic function = Range of the exponential function = $(0, \infty)$
- Range of the logarithmic function = Domain of the exponential function = $(-\infty, \infty)$

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

Domain: $0 < x < \infty$ Range: $-\infty < y < \infty$

Properties of the Logarithmic Function $f(x) = \log_a x$; $a > 0$, $a \neq 1$

- The domain is the set of positive real numbers, or $(0, \infty)$ using interval notation; the range is the set of all real numbers, or $(-\infty, \infty)$ using interval notation.
- The x-intercept of the graph is 1. There is no y-intercept.
- The y-axis $(x = 0)$ is a vertical asymptote of the graph of f.
- A logarithmic function is decreasing if $0 < a < 1$ and is increasing if $a > 1$.
- The graph of f contains the points $(1, 0)$, $(a, 1)$, and $\left(\frac{1}{a}, -1\right)$.
- The graph is smooth and continuous, with no corners or gaps.

$$y = \ln x \quad \text{if and only if} \quad x = e^y \qquad \textbf{(1)}$$

$$y = \log x \quad \text{if and only if} \quad x = 10^y$$

SUMMARY

Properties of the Logarithmic Function

$f(x) = \log_a x, \quad a > 1$

($y = \log_a x$ if and only if $x = a^y$)

- Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$
- x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); increasing; one-to-one
- The graph of f is smooth and continuous. It contains the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.
- See Figure 39(a) for a typical graph.

$f(x) = \log_a x, \quad 0 < a < 1$

($y = \log_a x$ if and only if $x = a^y$)

- Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$
- x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); decreasing; one-to-one
- The graph of f is smooth and continuous. It contains the points $(a, 1)$, $(1, 0)$, and $\left(\frac{1}{a}, -1\right)$.
- See Figure 39(b) for a typical graph.

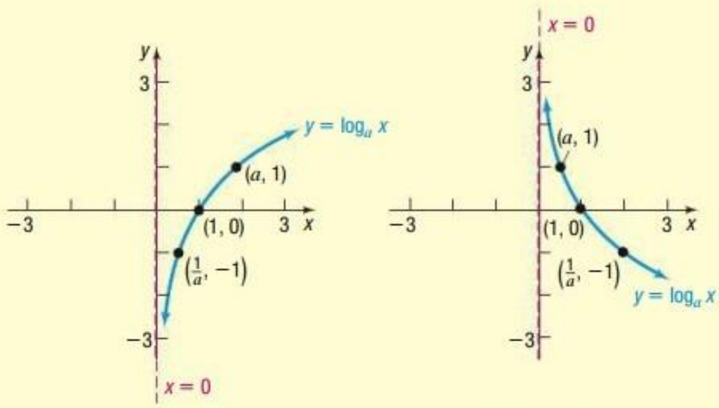

Figure 39 (a) $a > 1$ (b) $0 < a < 1$

$$\log_a 1 = 0 \qquad \log_a a = 1$$

THEOREM Properties of Logarithms

In these properties, M and a are positive real numbers, $a \neq 1$, and r is any real number.

- The number $\log_a M$ is the exponent to which a must be raised to obtain M. That is,

$$a^{\log_a M} = M \tag{1}$$

- The logarithm with base a of a raised to a power equals that power. That is,

$$\log_a a^r = r \tag{2}$$

THEOREM Properties of Logarithms

In these properties, M, N, and a are positive real numbers, $a \neq 1$, and r is any real number.

The Log of a Product Equals the Sum of the Logs

$$\log_a(MN) = \log_a M + \log_a N \tag{3}$$

The Log of a Quotient Equals the Difference of the Logs

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N \tag{4}$$

The Log of a Power Equals the Product of the Power and the Log

$$\log_a M^r = r \log_a M \tag{5}$$

$$a^r = e^{r \ln a} \tag{6}$$

THEOREM Properties of Logarithms

In these properties, M, N, and a are positive real numbers, $a \neq 1$.

- If $M = N$, then $\log_a M = \log_a N$. **(7)**
- If $\log_a M = \log_a N$, then $M = N$. **(8)**

THEOREM Change-of-Base Formula

If $a \neq 1$, $b \neq 1$, and M are positive real numbers, then

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \textbf{(9)}$$

$$\log_a M = \frac{\log M}{\log a} \quad \text{and} \quad \log_a M = \frac{\ln M}{\ln a}$$

SUMMARY

Properties of Logarithms

In the list that follows, a, b, M, N, and r are real numbers. Also, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$, $M > 0$, and $N > 0$.

Definition $y = \log_a x$ if and only if $x = a^y$

Properties of logarithms

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a M^r = r \log_a M$
- $a^{\log_a M} = M$
- $\log_a a^r = r$
- $a^r = e^{r \ln a}$
- $\log_a(MN) = \log_a M + \log_a N$
- $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$
- If $M = N$, then $\log_a M = \log_a N$.
- If $\log_a M = \log_a N$, then $M = N$.

Change-of-Base Formula $\log_a M = \frac{\log_b M}{\log_b a}$

Courtesy of **George Hartas**
Resource: Algebra & Trigonometry, 11th Edition, Michael Sullivan, 2020, Pearson Education