MAT 136 - Business Calculus - Quick Notes

Last Updated: 4/3/16

Chapter 2 – Applications of Differentiation

Section 2.1 – Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THE FIRST-DERIVATIVE TEST (FDT) FOR RELATIVE EXTREMA – Pg. 203

1. If $f'(x) < 0$ to the left of *critical value (cv)* and $f'(x) > 0$ to its right, have **relative min.**
2. If $f'(x) > 0$ to the left of *cv* and $f'(x) < 0$ to its right, have **relative max.**
3. If have same sign to the left of *cv* and to its right, neither a relative min nor max.

FIND RELATIVE EXTREMA THEN SKETCH GRAPH – Pg. 206

Note: Endpoints of a closed interval $[a, b]$ can be absolute extrema but *not* relative extrema.

1. Find *critical values* (*cv's*):
    a. Find $f'(x)$
    b. Set $f'(x) = 0$ to find *cv's.*
    c. Find where $f'(x)$ is undefined (but $f(x)$ is defined) to get additional *cv's.*
        i. (Polynomial functions are defined for all real numbers.)
    d. Find *y*-value for each *cv* by substituting *cv*'s into $f(x)$ to use for graphing later.
    e. Note: *cv's* do not guarantee relative extrema; they are candidates.
2. Use *cv*'s to divide *x*-axis number line into intervals.
    a. Choose a test value in each interval.
3. Substitute test values into $f'(x)$ to determine sign of result ($+ \; or \; -$) in each interval.
    a. Use FDT criteria to determine any relative extrema.
4. Graph the function:
    a. Plot points already found in Step 1d above.
    b. Plot additional points using table of values:
        i. Choose a number for *x* to find its *y*-value. Or choose a number for *y* to find its *x*-value. Substitute numbers into $f(x)$
        ii. Plot enough points to fill out graph.

<u>Section 2.2 – Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs</u>

<u>A TEST FOR CONCAVITY</u> – Pg. 216

1. If $f''(x) > 0$, graph is *concave up.* $f'(x)$ is *increasing,* $f(x)$ is *turning up.*
2. If $f''(x) < 0$, graph is *concave down.* $f'(x)$ is *decreasing,* $f(x)$ is *turning down.*

<u>THE SECOND-DERIVATIVE TEST (SDT) FOR RELATIVE EXTREMA</u> – Pg. 218

1. $f(c)$ is a **relative min** if $f''(c) > 0$
2. $f(c)$ is a **relative max** if $f''(c) < 0$
3. If $f''(c) = 0$, use FDT to determine relative extremum.

<u>Note</u>: the *critical value c* is substituted into $f''(x)$ to determine the sign for SDT.

<u>FINDING POINTS OF INFLECTION (POI)</u> – Pg. 221

<u>Note</u>: POI is a change in direction of concavity.

1. Find *possible points of inflection (PPOIs)* at x_0
   a. Find where $f''(x_0) = 0$ and $f''(x_0)$ *does not exist.*
   b. Find *y*-value of PPOI's by substituting x_0's into $f(x)$ to use for graphing later.
2. Determine whether the PPOIs are actually POIs:
   a. Use x_0's to divide *x*-axis number line into intervals.
   b. Choose a test value in each interval.
   c. Substitute test values into $f''(x)$ to determine sign of result $(+ \; or \; -)$ in each interval.
      i. If sign is different to the left of x_0 than to its right, have **POI.**
      ii. If sign is same to the left of x_0 as to its right, no POI.
      iii. Find *y*-value of POI's by substituting x_0's into $f(x)$ to use for graphing later.

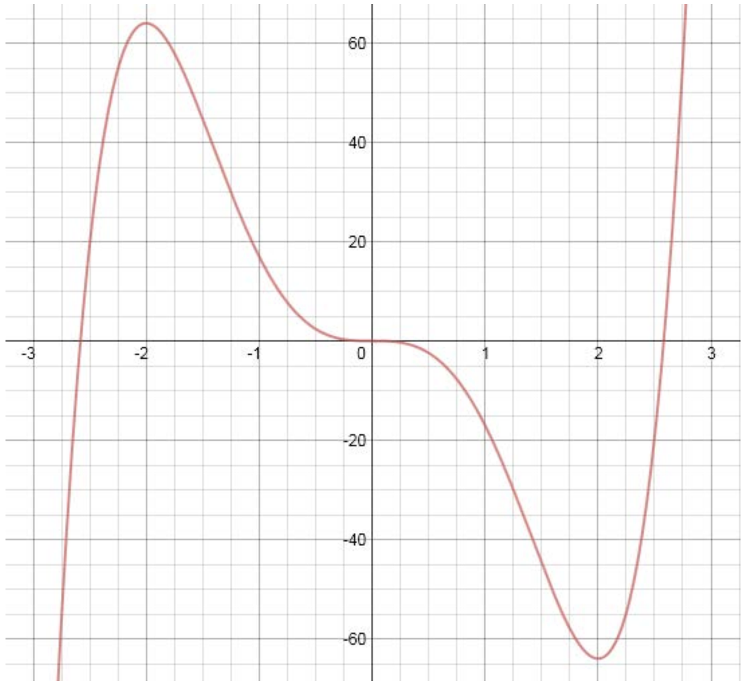
STRATEGY FOR SKETCHING GRAPHS – POLYNOMIAL FUNCTIONS – Pg. 222

1. *Derivatives* and *domain:*
    a. Find $f'(x)$
    b. Find $f''(x)$
    c. Find domain of $f(x)$. (Domain of polynomial functions is all real numbers.)
2. *Critical values* (*cv's*) of $f(x)$
    a. Set $f'(x) = 0$ to find *cv's.*
    b. Find where $f'(x)$ is undefined (but $f(x)$ is defined) to get additional *cv's.*
        i. (Polynomial functions are defined for all real numbers.)
    c. Find *y*-value for each *cv* by substituting *cv*'s into $f(x)$ to use for graphing later.
    d. Note: *cv's* do not guarantee relative extrema; they are candidates.
3. *Increasing / decreasing;* and *relative extrema:*
    a. Substitute *cv's* into $f''(x)$ for SDT.
    b. If $f''(c) < 0$, have **relative max**
        i. $f(x)$ is *increasing* to the left of *cv* and *decreasing* to its right. List interval.
    c. If $f''(c) > 0$, have **relative min**
        i. $f(x)$ is *decreasing* to the left of *cv* and *increasing* to its right. List interval.
    d. If $f''(c) = 0$, must use FDT to see if have relative extrema.
4. *Points of Inflection (POIs):*
    Note: POI is a change in direction of concavity.
    a. Find *possible points of inflection (PPOIs)* at x_0
        i. Find where $f''(x_0) = 0$ and $f''(x_0)$ *does not exist.*
        ii. Find *y*-value of PPOI's by substituting x_0's into $f(x)$ to use for graphing later.
    b. Determine whether the PPOIs are actually POIs:
        i. Use x_0's to divide *x*-axis number line into intervals.
        ii. Choose a test value in each interval.
        iii. Substitute test values into $f''(x)$ to determine sign of result ($+ \; or \; -$) in each interval.
            1. If sign is different to the left of x_0 than to its right, have **POI.**
            2. If sign is same to the left of x_0 as to its right, no POI.
            3. Find *y*-value of POI's by substituting x_0's into $f(x)$ to use for graphing later.
5. *Concavity:*
    a. List intervals of concavity using the PPOI *x*-axis number line in Step 4.
    b. If result in Step 4 was:
        i. $f''(x) > 0$, graph is *concave up* on that PPOI interval.
        ii. $f''(x) < 0$, graph is *concave down* on that PPOI interval.
6. *Sketch the graph:*
    a. Plot points already found in Step 2 and Step 4 above.
    b. Plot additional points using table of values:
        i. Choose a number for *x* to find its *y*-value. Or choose a number for *y* to find its *x*-value. Substitute numbers into $f(x)$
        ii. Plot enough points to fill out graph.
    c. Use the **intervals table** on next page to help sketch graphs of polynomial functions.

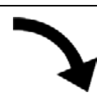
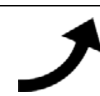
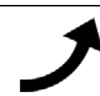
Example: The polynomial function below is sketched using the **intervals table** below.

$f(x) = 3x^5 - 20x^3$

Graph below courtesy of [Desmos Graphing Calculator]



Intervals Table for Sketching Graphs – Polynomial Functions *(Example)*

Interval Line	≈ -2.6	-2	≈ -1.4	0	$\approx +1.4$	$+2$	$\approx +2.6$	
	$(-\infty, \approx -2.6)$	$(\approx -2.6, -2)$	$(-2, \approx -1.4)$	$(\approx -1.4, 0)$	$(0, \approx +1.4)$	$(\approx +1.4, +2)$	$(+2, \approx +2.6)$	$(\approx +2.6, +\infty)$
Test Values x	$x = -3$	$x = -2.5$	$x = -1.5$	$x = -1$	$x = +1$	$x = +1.5$	$x = +2.5$	$x = +3$
$f'(x)$ Sign	+	+	−	−	−	−	+	+
$f''(x)$ Sign	−	−	−	+	−	+	+	+
Increasing / Decreasing	Increasing	Increasing	Decreasing	Decreasing	Decreasing	Decreasing	Increasing	Increasing
Concave up / Down	Concave Down	Concave Down	Concave Down	Concave Up	Concave Down	Concave Up	Concave Up	Concave Up
Graph Shape								

Notes:

- The values in the table above are from the example problem. Notice that the last row of table, Graph Shape, agrees with the graph image for all intervals.
- Partition the interval line with *x*-values from: *x*-intercepts, critical values, PPOI's.
  - List the *x*-values in numerical order from left to right.
  - In example above, the *x*-values are:
    - *x*-intercepts: ≈ -2.6, $\approx +2.6$
    - Critical values: –2, +2
    - PPOI's: ≈ -1.4, 0, $\approx +1.4$
- Substitute test values *x* into $f'(x)$ and $f''(x)$ to calculate the signs.
  - Then based on the signs, determine if function is increasing/decreasing and concavity.
  - Finally for each interval, sketch a graph shape based on increasing/decreasing and concavity.

Section 2.3 – Graph Sketching: Asymptotes and Rational Functions

VERTICAL ASYMPTOTE (VA) – Pg. 235

$$\lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty$$

Notes:

- A rational function may or may not have VA. Will not have VA if denominator will never be 0.
- If original rational function is already simplified (cannot factor), then if *a* makes denominator 0, have VA at *x* = *a*.
- If factored and cancelled a factor of the rational function, there is no VA at *a* that made denominator 0. Instead, it is called a "point of discontinuity," a "hole."
- Graph of rational function *never* crosses VA.

HORIZONTAL ASYMPTOTE (HA) – Pg. 237

$\lim_{x \to -\infty} f(x) = b$

$\lim_{x \to \infty} f(x) = b$

Notes:

- HA occurs when degree of numerator $\leq$ degree of denominator.
  - When degree of numerator = degree of denominator, HA is the leading coefficients of numerator and denominator.
  - When degree of numerator < degree of denominator, HA is the *x*-axis ($y = 0$).
- Graph of rational function *may or may not* cross an HA.
- Graph of rational function can have at most 2 HA's.

SLANT ASYMPTOTE (OBLIQUE ASYMPTOTE) – Pg. 239

Notes:

- Slant Asymptote occurs when degree of numerator is *exactly 1 more* than the degree of denominator.
- To find the *linear* slant asymptote, divide the numerator by the denominator using long division. The quotient is the slant asymptote.
- Graph of rational function *can* cross a slant asymptote.
- A rational function will have either an HA or a slant asymptote, but *not both.*

INTERCEPTS – Pg. 240

Notes:

- ***x*-Intercepts:**
  - To find *x*-intercepts of rational function, set $f(x) = 0$.
  - The *x*-intercepts occur when the numerator is zero and the denominator is not zero.
  - Factor numerator and denominator.
  - Set numerator = 0 and solve for *x*.
  - Check that these *x*-values do not make the denominator 0 also. If any do, they are not *x*-intercepts.
- ***y*-Intercept:**
  - To find *y*-intercept of rational function, set $f(0)$.

<u>STRATEGY FOR SKETCHING GRAPHS – RATIONAL FUNCTIONS</u> – Pg. 241

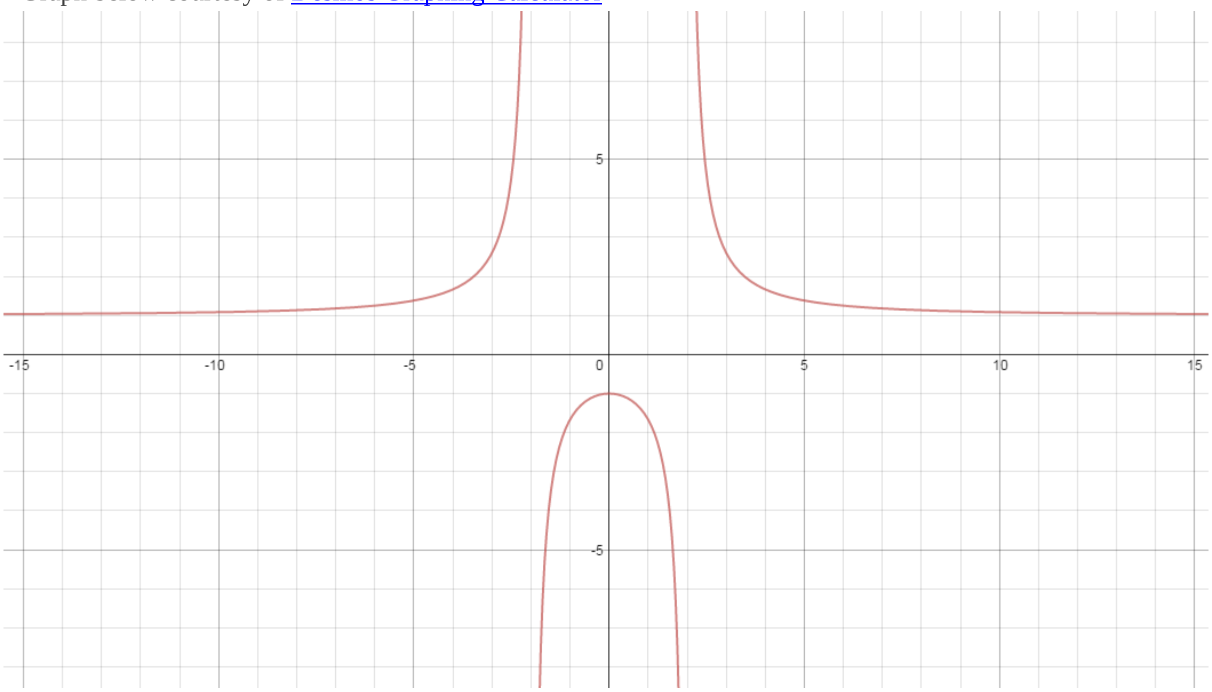
1. *Intercepts:*
    a. Find the *x*-intercepts and the *y*-intercept.
2. *Asymptotes:*
    a. Find any VA's, HA's, or slant asymptotes.
3. *Derivatives* and *domain:*
    a. Find $f'(x)$
    b. Find $f''(x)$
    c. Find domain of $f(x)$. List *x*-values that make denominator = 0 (from VA step).
4. *Critical values* (*cv's*) of $f(x)$
    a. Set $f'(x) = 0$ to find *cv's*. Set numerator = 0.
    b. Find where $f'(x)$ is undefined (but $f(x)$ is defined) to get additional *cv's*.
    c. Find *y*-value for each *cv* by substituting *cv*'s into $f(x)$ to use for graphing later.
    d. <u>Note</u>: *cv's* do not guarantee relative extrema; they are candidates.
5. *Increasing / decreasing;* and *relative extrema:*
    a. Substitute *cv's* into $f''(x)$ for SDT.
    b. If $f''(c) < 0$, have **relative max**
        i. $f(x)$ is *increasing* to the left of *cv* and *decreasing* to its right. List interval.
    c. If $f''(c) > 0$, have **relative min**
        i. $f(x)$ is *decreasing* to the left of *cv* and *increasing* to its right. List interval.
    d. If $f''(c) = 0$, must use FDT to see if have relative extrema.
6. *Points of Inflection (POIs):*
   <u>Note</u>: POI is a change in direction of concavity.
    a. Find *possible points of inflection (PPOIs)* at x_0
        i. Find where $f''(x_0) = 0$ and $f''(x_0)$ *does not exist.*
            1. If x_0 does not exist at $f(x)$, not a PPOI.
        ii. Find *y*-value of PPOI's by substituting x_0's into $f(x)$ to use for graphing later.
    b. Determine whether the PPOIs are actually POIs:
        i. Use x_0's to divide *x*-axis number line into intervals.
        ii. Choose a test value in each interval.
        iii. Substitute test values into $f''(x)$ to determine sign of result $(+ \; or \; -)$ in each interval.
            1. If sign is different to the left of x_0 than to its right, have **POI.**
            2. If sign is same to the left of x_0 as to its right, no POI.
            3. Find *y*-value of POI's by substituting x_0's into $f(x)$ to use for graphing later.
7. *Concavity:*
    a. List intervals of concavity using the PPOI *x*-axis number line in Step 6.
    b. If result in Step 6 was:
        i. $f''(x) > 0$, graph is *concave up* on that PPOI interval.
        ii. $f''(x) < 0$, graph is *concave down* on that PPOI interval.
8. *Sketch the graph:*

a. Plot points already found in Step 4 and Step 6 above.
b. Plot additional points using table of values:
   i. Choose a number for *x* to find its *y*-value. Or choose a number for *y* to find its *x*-value. Substitute numbers into $f(x)$
   ii. Plot enough points to fill out graph.
c. Use the **intervals table** on next page to help sketch graphs of rational functions.

Example: The rational function below is sketched using the **intervals table** on next page.

$$f(x) = \frac{x^2 + 4}{x^2 - 4}$$

Graph below courtesy of [Desmos Graphing Calculator]



Intervals Table for Sketching Graphs – Rational Functions *(Example)*

Interval Line	$(-\infty, -2)$	-2 $(-2, 0)$	0 $(0, +2)$	$+2$ $(+2, +\infty)$
Test Values x	$x = -10$	$x = -1$	$x = +1$	$x = +10$
$f(x)$ Value	$+1.08 > +1$ (**above HA**)	--	--	$+1.08 > +1$ (**above HA**)
$f'(x)$ Sign	$\frac{+}{+} = +$	$\frac{+}{+} = +$	$\frac{-}{+} = -$	$\frac{-}{+} = -$
$f''(x)$ Sign	$\frac{+}{+} = +$	$\frac{+}{-} = -$	$\frac{+}{-} = -$	$\frac{+}{+} = +$
Increasing / Decreasing	**Increasing**	**Increasing**	**Decreasing**	**Decreasing**
Concave up / Down	**Concave Up**	**Concave Down**	**Concave Down**	**Concave Up**
Graph Shape				

Notes:

- The values in the table are from the example problem on previous page. Notice that the last row of table, Graph Shape, agrees with the graph image for all intervals.
- Partition the interval line with *x*-values from: *x*-intercepts, VA's, critical values, PPOI's.
  - List the *x*-values in numerical order from left to right.
  - In example above, the *x*-values are –2 (VA), 0 (*cv*), and +2 (VA).
- The $f(x)$ Value row is used to determine the end behavior of the horizontal asymptote (HA).
  - The test values of $x = -10$ and $x = +10$ both result in the $f(x)$ value of $+1.08$, which is greater than the HA of $y = 1$. This means that the end behavior of the rational function places the graph **above the HA** at both ends on the *x*-axis.
  - Choose fairly large *x*-values to test for end behavior of HA.
  - The "inside" intervals for $f(x)$ do not need to be computed and have "--" in the table cell.
- Substitute test values x into $f'(x)$ and $f''(x)$ to calculate the signs.
  - Then based on the signs, determine if function is increasing/decreasing and concavity.
  - Finally for each interval, sketch a graph shape based on increasing/decreasing and concavity.
  - *Note:* This $\frac{-}{+} = -$ indicates the sign of the numerator, denominator, and the overall sign.
- To determine end behavior of a vertical asymptote, take the limit to the left and to the right of the VA.
  - If the limit is negative, the graph is decreasing on that side of the VA.
  - If the limit is positive, the graph is increasing on that side of the VA.
  - *Note:* This is not shown in the table above.

Section 2.4 – Using Derivatives to Find Absolute Maximum and Minimum Values

MAXIMUM-MINIMUM PINCIPLE 1 – Pg. 251

For a *closed* interval $[a, b]$.

1. Find $f'(x)$
2. Find *critical values* (*cv's*) in $[a, b]$
   a. Find where $f'(c) = 0$ and $f'(c)$ *does not exist.*
   b. A *cv* must be within interval $[a, b]$ to be considered for a max or min.
3. List values from Step 2 and the endpoints of the interval $[a, b]$.
4. Evaluate $f(x)$ for each value in Step 3.
   a. Largest value is **absolute max**
   b. Smallest value is **absolute min**

Notes:

- Absolute max and min values *can* occur at more than one point.
- Endpoints of a closed interval $[a, b]$ can be absolute extrema but *not* relative extrema.

MAXIMUM-MINIMUM PINCIPLE 2 – Pg. 253

For *exactly one cv* in interval *I*.

1. Find $f'(x)$
   a. $f'(x) = 0$ produces exactly one *cv*.
2. Find $f''(x)$
   a. If $f''(c) < 0$, $f(c)$ is an **absolute max**
   b. If $f''(c) > 0$, $f(c)$ is a **relative min**
   c. If $f''(c) = 0$, use Maximum-Minimum Principle 1; or we must know more about the behavior of the function to declare an absolute max or min.

Notes:

- The *critical value c* is substituted into $f''(x)$ to determine the sign for SDT.
- If the endpoints are not listed, they are considered to be $(-\infty, \infty)$.

A STRATEGY FOR FINDING ABSOLUTE MAXIMUM AND MINIMUM VALUES – Pg. 255

1. Find $f'(x)$
2. Find critical values.
   a. Find where $f'(c) = 0$ and $f'(c)$ *does not exist.*
3. If closed interval $[a, b]$ and:
   a. There is *more than one* critical value, use Maximum-Minimum Principle 1.
   b. There is *exactly one* critical value, use either Maximum-Minimum Principle 1 or Maximum-Minimum Principle 2.
      i. If it is easy to find $f''(x)$, use Maximum-Minimum Principle 2.
4. If open interval $(-\infty, \infty)$, $(0, \infty)$, or (a, b) and:
   a. There is *only one* critical value, use Maximum-Minimum Principle 2.
      i. Result will either be a max or min.
   b. There is *more than one* critical value, beyond scope of this course.

Section 2.5 – Maximum-Minimum Problems; Business and Economics Applications

A STRATEGY FOR SOLVING MAXIMUM-MINIMUM PROBLEMS – (Pg. 263)

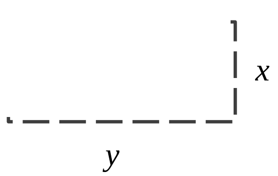
1. Read problem carefully. Make a drawing if it makes sense to do so.
2. List variables and constants. Note unit of measure. Label measurements on drawing.
3. Translate verbiage to an *equation* with the quantity *Q* to be minimized or maximized. Represent *Q* in terms of the variables listed in Step 2.
4. Express *Q* as a function of one variable. Determine max or min values and the points where they occur.

EX: 1 MAXIMIZE AREA – (Pg. 262)

Problem: Rectangular area needs to be fenced off only at 2 adjacent sides and maximized.

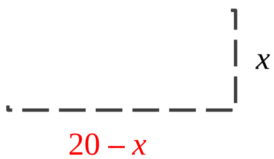
STEP 1 – *"Make a drawing."*

- Let *x* be the dimensions of one side; (*width* or smaller side).
- Let *y* be the dimensions of other side, (*length* or larger side).



STEP 2 – *"List variables and constants, unit of measure. Label measurements on drawing."*

- Have 20 *feet* of fencing available.
- Since $x + y = 20$, solve for y to have the 2 dimensions in terms of just one variable, x. So the length (larger side) becomes $y = 20 – x$. Relabel the y side of drawing as $20 – x$.



STEP 3 – *"Translate to an equation with the quantity Q to be maximized. Represent Q in terms of the variables listed in Step 2."*

- Since trying to find maximum area, need to use the Area Formula: $A = lw$
- Substitute problem variables ($20 – x$ and x) into Area Formula:

  $A = lw$ – Area Formula is the *starting equation*

  $A = (20 - x)x$ – Area Formula after substituting problem variables

STEP 4 – *"Express Q as a function of one variable. Determine max or min values and the points where they occur."*

$A(x) = 20x - x^2$ – Area Formula to be maximized, after multiplying and simplifying.
– Domain is $(0, 20)$ since dimensions in feet cannot be < 0, 0, 20, or > 20.

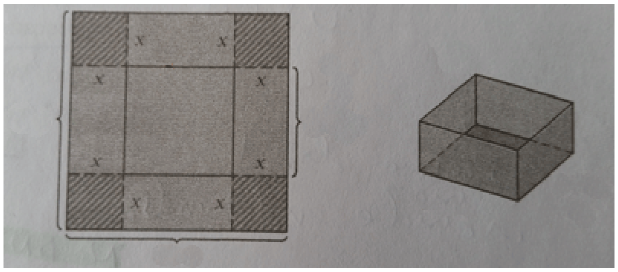
- Find $A'(x) = 20 - 2x$
- Set $A'(x) = 0$ to find *critical values cv's.*
  - $20 - 2x = 0$
  - $x = 10$
- Since there's only 1 *cv*, use *Maximum-Minimum Principle 2* (second derivative) to determine if this *cv* is a maximum.
  - $A''(x) = -2$ – This is a negative constant so the *cv* $x = 10$ is a maximum.
- Conclusion:
  - $x = 10$ represents the dimension of 10 feet for side x.
  - Substitute 10 for x into $y = 20 – x$ ⇨ $y = 20 – 10$ ⇨ $y = 10$ which represents the dimension of 10 feet for side y.
  - Then find area by $A = lw$ ⇨ $A = (10)(10)$ ⇨ $A = 100$
  - The dimensions of the two sides should each be 10 feet to give a maximum area of 100ft^2.
- Video Related To This Example: https://www.youtube.com/watch?v=jpFquCszOX0

EX: 2 MAXIMIZE VOLUME – (Pg. 264)

Problem: A cardboard 8 in. by 8 in. needs to have a square cut out from corners so the sides can be folded up to make an open-top box. What dimension will yield maximum volume? What is the maximum volume?

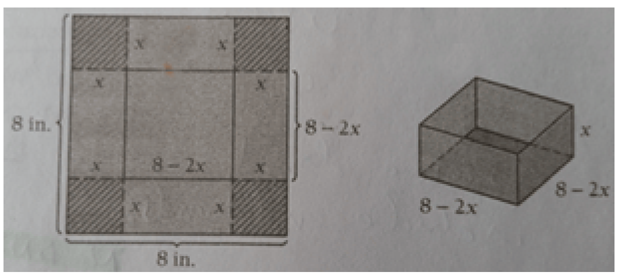
STEP 1 – *"Make a drawing."*

- Let x be the dimensions of each square corner to be cut.



STEP 2 – *"List variables and constants, unit of measure. Label measurements on drawing."*

- Since original square is 8 in. by 8 in., after the four corner squares (x's) are cut out, the dimensions of the sides will be $(8 - 2x)$ in. by $(8 - 2x)$ in. The part labeled x is going to be the height of the box after the sides are folded up.



STEP 3 – *"Translate to an equation with the quantity Q to be maximized. Represent Q in terms of the variables listed in Step 2."*

- Since trying to find maximum volume, need to use the Volume Formula: $V = lwh$
- Substitute problem variables of length, width, height $(8 - 2x)$, $(8 - 2x)$, x into Volume Formula:

  $V = lwh$ – Volume Formula is the *starting equation*

  $V = (8 - 2x)(8 - 2x)x$ – Volume Formula after substituting problem variables

<u>STEP 4</u> – *"Express Q as a function of one variable. Determine max or min values and the points where they occur."*

$V(x) = 4x^3 - 32x^2 + 64x$ – Volume Formula to be maximized, after multiplying and simplifying.

– Domain: since a side 8 – 2x > 0, each square corner x < 4. Therefore (0, 4).

- Find $V'(x) = 12x^2 - 64x + 64$
- Set $V'(x) = 0$ to find *critical values cv's.*
    - $12x^2 - 64x + 64 = 0$
    - $x = \frac{4}{3}, 4$
- Because the *cv* 4 is not in the domain (0, 4), have only one cv, $\frac{4}{3}$.
- Since there's only 1 *cv*, use *Maximum-Minimum Principle 2* (second derivative) to determine if this *cv* is a maximum.
    - $V''(x) = 24x - 64$ – Find second derivative.
    - $V''\left(\frac{4}{3}\right) = 24\left(\frac{4}{3}\right) - 64$ ⇨ 32 – 64 < 0 – Substitute $\frac{4}{3}$ into V'' and result is negative, therefore $\frac{4}{3}$ is a maximum.
- Conclusion:
    - $x = \frac{4}{3}$ represents the $\frac{4}{3}$ in. of the corner square, *x*, that needs to be cut out to maximize the box's volume.
    - After the $\frac{4}{3}$ in. has been cut out from each corner and the sides have been folded up, the dimensions of the two equal sides of the box will be: $8 - 2x$ ⇨ $8 - 2\left(\frac{4}{3}\right)$ ⇨ $5\frac{1}{3}$ in.
    - The height of the box is $\frac{4}{3}$ in. which is obtained when the *x* is cut out (the $\frac{4}{3}$ in.) and the remaining part of the side is folded up (the $5\frac{1}{3}$ in.).
    - The maximum volume of the box is:

      $V(x) = 4x^3 - 32x^2 + 64x$ ⇨ $V\left(\frac{4}{3}\right) = 4\left(\frac{4}{3}\right)^3 - 32\left(\frac{4}{3}\right)^2 + 64\left(\frac{4}{3}\right)$ ⇨ $37\frac{25}{27} in^3$
- Video Related To This Example: https://www.youtube.com/watch?v=wCVuupUndb0

Ex: 3 Minimize Surface Area (Pg. 265) – https://www.youtube.com/watch?v=T1m8BN7UbOg
Ex: 4 Maximize Revenue and Profit (Pg. 267) – https://www.youtube.com/watch?v=-TzmLSQZFoI

EXERCISE SET 2.5, # 31: MAXIMIZE REVENUE *(Determining a Ticket Price)* – (Pg. 274)
Similar to **Ex: 5 Maximize Revenue** (Pg. 268)

Problem: A university is trying to determine what price to charge for tickets to football games. At a price of $18 per ticket, attendance averages 40,000 people per game. Every decrease of $3 adds 10,000 people to the average number. Every person at the game spends an average of $4.50 on concessions. What price per ticket should be charged in order to maximize revenue? How many people will attend at that price?

STEP 1 – *"Make a drawing."*

- Not applicable.

STEP 2 – *"List variables and constants, unit of measure."*

- Revenue $= R(x) =$ Quantity • Price
- Two sources of revenue in this type of problem: ticket revenue + concession revenue.
- *Ticket revenue:*
    - Ticket revenue with "normal" operation (average attendance with normal ticket price):
        - Quantity = 40,000 people
        - Price = $18 ticket price
        - $R(x) =$ (40,000 people) ($18)
    - Ticket revenue with problem constraints (restrictions on "normal" operation):
        - Quantity = 40,000 people + 10,000 people (for every decrease of $3 per ticket)
            - (40,000 + 10,000*x*)
            - Note: The *x* is the number of times the ticket is reduced by $3. The *x* represents some multiple of the number of times the ticket is reduced by $3. The *x* is a multiplier with the change in attendance.
        - Price = $18 and each occurrence that the ticket is decreased by $3.
            - (18 – 3*x*)
            - Note: As noted above, *x* is the number of times the ticket is reduced by $3. The *x* represents some multiple of the number of times the ticket is reduced by $3.
- *Concession revenue:*
    - Concession revenue with "normal" operation (average attendance with normal concession spending):
        - Quantity = 40,000 people
        - Price = $4.50 concession spending per person
        - $R(x) =$ (40,000 people) ($4.50)
    - Concession revenue with problem constraints (restrictions on "normal" operation):
        - Quantity = 40,000 people + 10,000 people (for every decrease of $3 per ticket)
        - Price = $4.50 concession spending per person *remains the same as "normal" operation.* The $4.50 is a constant.
        - (40,000 + 10,000*x*) (4.50)
        - Note: The quantity "attendance" (40,000 + 10,000*x*) is the same for concession revenue as it is for ticket revenue.

STEP 3 – *"Translate to an equation with the quantity R(x) to be maximized. Represent R(x) in terms of the variables listed in Step 2."*

Find total revenue function:

$R(x)$ = Total revenue = (Ticket revenue) + (Concession revenue)

$R(x)$ = (Number of people) (Ticket price) + (Number of people) (Concession spending)

$R(x) = (40{,}000 + 10{,}000x)(18 - 3x) + (40{,}000 + 10{,}000x)(4.50)$

$R(x) = 720{,}000 - 120{,}000x + 180{,}000x - 30{,}000x^2 + 180{,}000 + 45{,}000x$

$R(x) = -30{,}000x^2 + 105{,}000x + 900{,}000$

Find $R'(x) = -60{,}000x + 105{,}000$

Set $R'(x) = 0$ to find *critical values cv's.*

$-60{,}000x + 105{,}000 = 0$

$x = 1.75$

Since there's only 1 *cv*, use *Maximum-Minimum Principle 2* (second derivative) to determine if this *cv* is a maximum.

$R''(x) = -60{,}000$ – Since second derivative is a negative constant, 1.75 is a maximum.

- Conclusion:
  - Price per ticket to charge:
    - $(18 - 3x)$
    - $(18 - 3 \cdot 1.75)$ – Substitute 1.75 for x
    - 12.75 – To maximize profit, the ticket price should be \$12.75
  - Attendance:
    - $(40{,}000 + 10{,}000x)$
    - $(40{,}000 + 10{,}000 \cdot 1.75)$ – Substitute 1.75 for x
    - 57,000 – At a ticket price of \$12.75, the amount of people that will attend per game will be 57,000.

Additional Resources for This Type of Problem

- Videos:
  - https://www.youtube.com/watch?v=vfIFLryA_DU
  - https://www.youtube.com/watch?v=WWkwunkQc-c
- How to Solve an Optimization Problem? (PDF) – By Dr. Mohammed Yahdi

Chapter 4 – Integration

Section 4.1 – Antidifferentiation

RULES OF ANTIDIFFERENTIATION – Pg. 391

Constant Rule

$$\int k\,dx = kx + C$$

Power Rule

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

Natural Logarithm Rule

$$\int \frac{1}{x} dx = \ln x + C, \quad x > 0$$

Exponential Rule (base e)

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0$$

PROPERTIES OF ANTIDIFFERENTIATION – Pg. 393

$$\int [cf(x)]dx = c\int f(x)dx$$

Section 4.2 – Antiderivatives as Areas

RIEMANN SUMMATION – Pg. 401-405

Area of Region Under the Curve

$$\sum_{i=1}^{n} f(x_i)\Delta x, \quad where\ \Delta x = \frac{b-a}{n}$$

STEPS FOR THE PROCESS OF RIEMANN SUMMATION – Pg. 405

1. Draw the graph of $f(x)$.
2. Subdivide the interval $[a, b]$ into *n* subintervals of equal width. Calculate the *width* of each rectangle by using the formula:
$$\Delta x = \frac{b-a}{n}$$
3. Construct rectangles above the subintervals such that the *top left corner* of each rectangle touches the graph.
4. Determine the *area of each rectangle.*
5. *Sum these areas* to arrive at an approximation for the total area under the curve.

DEFINITE INTEGRAL – Pg. 405

A *definite integral* is the limit as $n \to \infty$ (equivalently, $\Delta x \to 0$) of the Riemann sum.

$$Exact\ Area = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

Section 4.3 – Area and Definite Integrals

DEFINITE INTEGRAL – Pg. 414

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where *a* is lower limit and *b* is higher limit.

THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS – Pg. 415

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx = F(b) - F(a)$$

Section 4.4 – Properties of Definite Integrals

ADDITIVE PROPERTY OF DEFINITE INTEGRALS – Pg. 425

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

For $a < b < c$

Useful for *piecewise* functions.

AREA OF REGION BOUNDED BY TWO GRAPHS – Pg. 428

$$\int_a^b [f(x) - g(x)]dx$$

Where $f(x) \geq g(x)$ over the interval $[a, b]$

STEPS TO FIND AREA OF REGION BOUNDED BY TWO GRAPHS

1. Sketch graphs to determine which one is the upper graph.
2. Set $f(x) = g(x)$ and solve for x. These x-values are the limits of integration.
3. Set up integral by substituting the appropriate functions as the $f(x)$ and $g(x)$ into the formula above.
4. Simplify $f(x) - g(x)$ to get one polynomial.
5. Integrate.

AVERAGE VALUE OF A FUNCTION – Pg. 431

$$y_{av} = \frac{1}{b-a}\int_a^b f(x)dx$$

Section 4.5 – Integration Techniques: Substitution

Power Rule

$$\int u^r du = \frac{u^{r+1}}{r+1} + C, \quad r \neq -1$$

Exponential Rule (base e)

$$\int e^u du = e^u + C$$

Natural Logarithm Rule

$$\int \frac{1}{u} du = \ln|u| + C, \quad or \qquad \int \frac{1}{u} du = \ln u + C, \quad if\ u > 0$$

STRATEGY FOR SUBSTITUTION – Pg. 442

1. Decide which rule is appropriate; see above.
   a. If *Power Rule*, let *u* be the base.
   b. If *Exponential Rule (base e)*, let *u* be the expression in the exponent.
   c. If *Natural Logarithm Rule*, let *u* be the denominator.
2. Determine *du*.
3. Ensure substitution accounts for all factors in integrand. May need to insert constants or make an extra substitution.
4. Integrate.
5. Reverse the substitution. If there are bounds, evaluate integral *after* substitution has been reversed.
6. Check by differentiating.

Chapter 5 – Applications of Integration

Section 5.1 – An Economics Application: Consumer Surplus and Producer Surplus

CONSUMER SURPLUS (DEMAND CURVE) – Pg. 475

Consumer surplus is defined for the point (Q, P) as

$$\int_0^Q D(x)dx - QP$$

Where,
Q is *Quantity*
P is *Price*
$D(x)$ is *total utility*
QP is *total cost*

Consumer surplus is the "pleasure received but did not pay for." Aka *Utility* or *U*.

PRODUCER SURPLUS (SUPPLY CURVE) – Pg. 476

Producer surplus is defined for the point (Q, P) as

$$QP - \int_0^Q S(x)dx$$

Where,
Q is *Quantity*
P is *Price*
QP is *revenue*
$S(x)$ is *total cost*

Producer surplus is a contribution to profit.

EQUILIBRIUM POINT (x_E, p_E) – Pg. 477

$$D(x) = S(x)$$

Where,
$x_E = Q$
$p_E = P$

Solve for *x* then substitute into either $D(x)$ or $S(x)$ to find *y*.

CONSUMER SURPLUS AT THE EQUILIBRIUM POINT – Pg. 478

$$\int_0^{x_E} D(x)dx - x_E p_E$$

Where,
$D(x)$ is *total utility*
$x_E p_E$ is *total cost*

PRODUCER SURPLUS AT THE EQUILIBRIUM POINT – Pg. 478

$$x_E p_E - \int_0^{x_E} S(x)dx$$

Where,
$x_E p_E$ is *revenue*
$S(x)$ is *total cost*